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Meir–Keeler-type conditions in abstract metric spaces[☆]

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ABSTRACT

Various Meir–Keeler-type conditions for mappings acting in abstract metric spaces are presented and their connections are discussed. Results about associated symmetric spaces, obtained in [S. Radenović, Z. Kadelburg, Quasi-contractions on symmetric and cone symmetric spaces, Banach J. Math. Anal. 5 (2011), 38–50] are used to show that the regularity condition for the underlying cone can be dropped in some fixed point results that have appeared recently.

1. Introduction and preliminaries

Meir–Keeler's result, proved in 1969 [1], plays a fundamental role in the fixed point theory for metric spaces and is still a matter of investigation in Nonlinear Analysis (see, e.g., [2–4]). Haghi and Rezapour [5] proved this important theorem in the setting of cone metric spaces, introduced by Huang and Zhang [6] (see also [7,8], as well as [9,10] for other approaches to abstract metric). We introduce in this work several Meir–Keeler-type conditions in the setting of such spaces and investigate connections among them. In particular, using results about associated symmetric spaces, obtained in [11], we show that the regularity condition for the underlying cone can be dropped in the result of Haghi and Rezapour.

We need the following definitions and results, consistent with [6,12,13].

Let E be a real Banach space with the zero vector θ. A subset K of E is called a *cone* if: (a) K is closed, non-empty and $K \neq \{\theta\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in K$ imply that $ax + by \in K$; (c) $K \cap (-K) = \{\theta\}$.

Given a cone K, we define the partial ordering $\preceq$ with respect to K by $x \preceq y$ if and only if $y - x \in K$. We shall write $x \ll y$ for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ stands for the interior of K and use $x \prec y$ for $x \preceq y$ and $x \neq y$.

If $\operatorname{int} K \neq \emptyset$, then K is called a *solid cone* [13].

The cone K in the Banach space E is called *normal* if there is a number $k > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq k\|y\|$ (the minimal such constant k is called the normal constant of K). Equivalently, the cone K is normal if

$$(\forall n)\ x_n \preceq y_n \preceq z_n \quad \text{and} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.1}$$

For details see [12].

The cone K in E is called *regular* if every increasing sequence in E which is bounded from above is convergent. Equivalently, the cone K is regular if every decreasing sequence in E which is bounded from below is convergent. Every regular cone is normal [12], but the converse is not true.

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Example 1.1 (*[13]*). 1° Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $K = \{x \in E : x(t) \geq 0\}$. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by (1.1) that K is a nonnormal cone.

2° Let $E = C_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty$ and K be as in the previous example. Then K is a normal cone, but it is not regular. Indeed, let $x_n(t) = -t^n$; then the sequence $\{x_n\}$ is increasing and bounded from above but $\|x_n\| = 1$ for all n, so $\lim_{n\to\infty} x_n$ does not exist.

Definition 1.2 (*[6,14]*). Let X be a non-empty set and E a Banach space with a cone K. Suppose that a mapping $d : X \times X \to E$ satisfies:

(d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The mapping d is called an *abstract metric* and (X, d) is called an *abstract metric space* (or a cone metric space [6], or a K-metric space [14]); we shall use the first mentioned term.

For examples of abstract metric spaces and definition of basic notions, in particular convergent and Cauchy sequences, completeness etc., we refer to [6,14] (see also [9,10]).

In what follows we assume that E is a real Banach space and that K is a normal and solid cone in E. The partial ordering induced by the cone K will be denoted by $\preceq$.

Let (X, d) be an abstract metric space over K. The following properties are valid, even in the case when the underlying cone is nonnormal.

(p_1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(p_2) If $\theta \preceq u \ll c$ for each c, $\theta \ll c$, then $u = \theta$.
(p_3) If $c \in \operatorname{int} K$, $\theta \preceq x_n$ and $x_n \to \theta$, then there exists $k \in \mathbb{N}$ such that for all $n > k$ we have $x_n \ll c$. (Note that the converse is not true if K is nonnormal. Indeed, in Example 1.1.1°, $x_n \not\to \theta$, but $x_n \ll c$ for n sufficiently large.)

For the space (X, d) one can construct a symmetric space (X, D) where "symmetric" $D : X \times X \to \mathbb{R}$ (in the sense of [11,15]) is given by $D(x, y) = \|d(x, y)\|$. Note that mappings with the same properties were also investigated under the name of "quasimetric" in [16–18].

Definition 1.3 (*[11]*). The space (X, D) is called the symmetric space associated with the abstract metric space (X, d).

Since the underlying cone is normal, the triangle inequality implies that the symmetric $D = \|d\|$ satisfies the condition

$$D(x, y) \leq k(D(x, z) + D(z, y)),$$

where $k \geq 1$ is the normal constant of K. Note that if $k = 1$, then (X, D) is a metric space, but in general it is not.

Now, for $x \in X$ and $\varepsilon > 0$ let $B_\varepsilon(x) = \{y \in X : D(y, x) < \varepsilon\}$. Let t_D be the topology on X generated by the balls of the form $B_\varepsilon(x)$, $x \in X$, $\varepsilon > 0$.

Theorem 1.4 (*[11]*). *Let (X, d) be an abstract metric space with a normal cone K and let D be the associated symmetric. Then topologies induced by d and D on X are the same, $t_d = t_D$.*

2. Meir–Keeler-type conditions for mappings in abstract metric spaces

Meir and Keeler [1] introduced a contractive-type condition (which they called the condition of weakly uniformly strict contraction). This condition was used by many authors to obtain various fixed point results (see, e.g., [2–4]).

Definition 2.1 (*[1]*). Let (X, d) be a metric space and let $f : X \to X$ be a selfmap. It is said that f has the property (MK) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in X$, $\varepsilon \leq d(x, y) < \varepsilon + \delta$ implies that $d(fx, fy) < \varepsilon$.

It is easy to see that in metric spaces property (MK) is equivalent to the following property
(**MK**$_1$) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in X$, $d(x, y) < \varepsilon + \delta$ implies that $d(fx, fy) < \varepsilon$.
A similar conclusion is valid in symmetric spaces (X, D).
Haghi and Rezapour [5] introduced a Meir–Keeler-type condition for mappings in abstract metric spaces. We shall consider several conditions of this type.

Definition 2.2. Let (X, d) be an abstract metric space and let $f : X \to X$ be a selfmap. We say that f has the property:
(**MKA**$_1$) [5] if for every $\theta \neq c \in K$ there exists $\theta \ll d$ such that for each $x, y \in X$, $d(x, y) \prec c + d$ implies $d(fx, fy) \prec c$.
(**MKA**$_2$) if for every $\theta \neq c \in K$ there exists $\theta \ll d$ such that for each $x, y \in X$, $c \preceq d(x, y) \prec c + d$ implies $d(fx, fy) \prec c$.
(**MKA**$_3$) if for every $\theta \neq c \in K$ there exists $\theta \ll d$ such that for each $x, y \in X$, $d(x, y) \prec c + d$ implies $d(fx, fy) \preceq c$.

It is clear that (MKA_1) implies (MKA_2).

Lemma 2.3. *If the cone K satisfies the condition*

$$\text{for each } a, b \in K \setminus \{\theta\} \text{ there exists } c \in K \setminus \{\theta\} \text{ such that } c \preceq a \text{ and } c \preceq b, \tag{2.1}$$

then property (MKA_2) implies property (MKA_1).

Proof. Let property (MKA_2) hold and let $\theta \prec c$ be given. For $x, y \in X$, we are looking for $\theta \ll d$ so that (MKA_1) holds. If $c \preceq d(x, y)$, then the condition is satisfied. Let $\alpha := d(x, y) \prec c$. Then, using (MKA_2), one can find $\theta \ll d_1$ so that $\alpha \preceq d(x, y) \prec \alpha + d_1$ implies $d(fx, fy) \prec \alpha \prec c$. Taking $d = d_1$ we have that $d(x, y) \prec \alpha + d_1 \prec c + d$ implies $d(fx, fy) \prec c$.

It remains to consider the case when c and $d(x, y)$ are incomparable. Denote by z an element of $K \setminus \{\theta\}$ satisfying $z \preceq c$ and $z \preceq d(x, y)$. Choose $\theta \ll d_2$ such that $z \preceq d(x, y) \prec z + d_2$ implies $d(fx, fy) \prec z$. Then, obviously, $d(x, y) \prec c + d$ implies $d(fx, fy) \prec c$ and we can take $d = d_2$ in condition (MKA_1). □

We do not know whether condition (2.1) can be omitted in the previous Lemma.

It is obvious that property (MKA_1) implies property (MKA_3). In the next example we show that the converse is not true.

Example 2.4. Let $X = A \cup B \cup C$, where $A = [-1, 0]$, $B = \{3n : n \in \mathbb{N}\}$, $C = \{3n + 1 + \frac{1}{3n} : n \in \mathbb{N}\}$ and let $f : X \to X$ be defined by $fx = 0$ for $x \in A \cup B$ and $fx = 1$ for $x \in C$. Let $E = \mathbb{R}^2$, $K = \{ (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$, and let $d(x, y) = (|x - y|, 0)$ for all $x, y \in X$. The mapping f satisfies property (MKA_3). Indeed, let $c = (c_1, c_2) \in K \setminus \{\theta\}$. If $c_1 \geq 1$, take $d = (\delta_1, \delta_2)$ with arbitrary $\delta_1 > 0$, $\delta_2 > 0$, and obtain that $d(fx, fy) = (|fx - fy|, 0) \preceq (1, 0) \preceq c$ for arbitrary $x, y \in X$. Suppose that $0 < c_1 < 1$ and take $\delta_1 = 1 - c_1$ and $\delta_2 > 0$ arbitrary. Then $d(x, y) \prec c + d = (c_1, c_2) + (1 - c_1, \delta_2) = (1, c_2 + \delta_2)$ is equivalent to $|x - y| < 1$, which is only possible if $x, y \in A$. So, $d(fx, fy) = (0, 0) \prec (c_1, c_2) = c$ and, *a fortiori*, $d(fx, fy) \preceq c$.

However, f does not satisfy property (MKA_1). If $c = (1, 0)$, then for each $\theta \ll d = (\delta_1, \delta_2)$ there is $n \in \mathbb{N}$ such that $\frac{1}{3n} < \delta_1$. Take $x = 3n \in B$, $y = 3n + 1 + \frac{1}{3n} \in C$. Then the condition $d(x, y) \prec c + d$ holds, since it is equivalent to $(|x - y|, 0) = (1 + \frac{1}{3n}, 0) \prec (1 + \delta_1, \delta_2)$. But, $d(fx, fy) = (|fx, fy|, 0) = (1, 0) = c$, and so $d(fx, fy)$ is not strictly smaller than c.

Lemma 2.5. *Let a selfmap f satisfy property (MKA_1) in an abstract metric space (X, d). Then it satisfies property (MK_1) in the associated symmetric space (X, D) (where $D(x, y) = \|d(x, y)\|$), i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in X$, $D(x, y) < \varepsilon + \delta$ implies $D(fx, fy) < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and $x, y \in X$. Since $\operatorname{int} K \neq \emptyset$, there exists $c \in \operatorname{int} K$ such that $k \cdot \|c\| < \varepsilon$. Using (MKA_1), for this c and for the chosen $x, y \in X$, one can find $\theta \ll d$ such that $d(x, y) \prec c + d$ implies $d(fx, fy) \prec c$. Using normality of the cone, we conclude that $\|d(x, y)\| \leq k\|c + d\| \leq k\|c\| + k\|d\|$ implies $\|d(fx, fy)\| \leq k\|c\|$. In other words, $D(x, y) < \varepsilon + \delta$ implies that $D(fx, fy) < \varepsilon$, where we have put $\delta = k\|d\|$. □

Remark 2.6. Condition (MKA_3) implies the condition

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(D(x, y) < \varepsilon + \delta \Rightarrow D(fx, fy) \leq \varepsilon) \tag{2.2}$$

which is equivalent with

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in X)(\varepsilon < D(x, y) < \varepsilon + \delta \Rightarrow D(fx, fy) \leq \varepsilon). \tag{2.3}$$

Lemma 2.7. *Let a selfmap f satisfy property (MKA_3) in an abstract metric space (X, d) with the associated symmetric space (X, D). Then f is d- and D-strict-contractive, i.e.,*

(a) $d(fx, fy) \prec d(x, y)$

(b) $D(fx, fy) < D(x, y)$,

for all $x, y \in X$, $x \neq y$.

Proof. (a) Condition (MKA_3) implies that

$$(\forall c > 0)(\exists d \in \operatorname{int} K)(\forall x, y \in X)(c \prec d(x, y) \prec c + d \Rightarrow d(fx, fy) \preceq c),$$

wherefrom property (a) follows.

(b) By Remark 2.6, property (MKA_3) implies condition (2.3) and the conclusion follows. □

3. Fixed point results for Meir–Keeler-type mappings in normal abstract metric spaces

Theorem 3.1. *Let (X, d) be a complete abstract metric space with a normal cone K and let $f : X \to X$ have the property (MKA_3). Then f has a unique fixed point u and for each $x_0 \in X$, the sequence $\{f^n x_0\}$ of Picard iterations converges to u.*

Proof. Let D be the cone symmetric associated with d (Definition 1.3). By Lemma 2.7, property (MKA_3) implies that $D(fx, fy) < D(x, y)$ for all $x, y \in X$, $x \neq y$. Let $x_0 \in X$ be arbitrary and consider the sequence $\{x_n\}$ defined by $x_n = f^n x_0$ for $n \geq 0$. If $x_m = x_{m+1}$ for some m, then x_m is a fixed point for f. Suppose that $x_m \neq x_{m+1}$ for all $m \in \mathbb{N}_0$. This means that $D(x_{n+1}, x_n) < D(x_n, x_{n-1})$ for $n \in \mathbb{N}$, i.e., $\{D(x_{n+1}, x_n)\}$ is a strictly decreasing sequence of positive real numbers. It converges to some $D^* \geq 0$.

Suppose that $D^* > 0$. Then, using Remark 2.6, there exists $\delta > 0$ such that for all $x, y \in X$, $D(x, y) < D^* + \delta$ implies $D(fx, fy) \leq D^*$. Choose $n_0 \in \mathbb{N}$ such that $D(x_{n+1}, x_n) < D^* + \delta$ for $n \geq n_0$, which implies that $D(x_{n+2}, x_{n+1}) = D(fx_{n+1}, fx_n) \leq D^*$, a contradiction. We conclude that $D^* = 0$, i.e., $D(x_{n+1}, x_n)$, and so also $d(x_{n+1}, x_n)$ converges to zero (in $\mathbb{R}$, resp. in K).

Let us prove now that $\{x_n\}$ is a Cauchy sequence. By Lemma 2.7(a), $d(fx, fy) \prec d(x, y)$ for all $x \neq y$, and $\{d(x_{n+1}, x_n)\}$ is a strictly decreasing sequence of vectors in the cone K. It converges to zero, as has just been proved. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $c \in \operatorname{int} K$ such that for each $n_0 \in \mathbb{N}$ there exist $n_1, n_2 \in \mathbb{N}$, $n_1, n_2 > n_0$, such that $d(x_{n_1}, x_{n_2}) \ll c$ does not hold. Property (MKA_3) implies that for each e, $\theta \ll e \ll c$, there exists $\theta \ll d$ such that for all $x, y \in X$, $d(x, y) \prec e + d$ implies $d(fx, fy) \preceq e$. Since $d(x_{n+1}, x_n) \to \theta$ when $n \to \infty$, for the given interior point d of the cone K there exists $n_3 \in \mathbb{N}$ such that $d(x_{n+1}, x_n) \ll d/2$ whenever $n \geq n_3$ (property (p_3)). Also, there exist natural numbers $k_1 > k_2 > n_3$ such that $d(x_{k_1}, x_{k_2}) \ll c$ does not hold. Now we have

$$d(x_{k_1-1}, x_{k_1+1}) \preceq d(x_{k_1-1}, x_{k_1}) + d(x_{k_1}, x_{k_1+1}) \ll \frac{d}{2} + \frac{d}{2} = d \prec e + d,$$

wherefrom, using (MKA_3), it follows that $d(x_{k_1}, x_{k_1+2}) \preceq e$. In the same way, $d(x_{k_1-1}, x_{k_1+2}) \preceq d(x_{k_1-1}, x_{k_1}) + d(x_{k_1}, x_{k_1+2}) \ll \frac{d}{2} + e \prec e + d$, implying that $d(x_{k_1}, x_{k_1+3}) \preceq e$. Continuing in this way, one obtains that $d(x_{k_1}, x_{k_2}) \preceq e$ and, *a fortiori*, $d(x_{k_1}, x_{k_2}) \ll c$ (property (p_1)), which is a contradiction.

We conclude that the sequence $\{x_n\}$, $x_n = f^n x_0$, is a Cauchy sequence.

The rest of the proof is standard. □

Remark 3.2. Our result is stronger then Proposition 2.1 in [5] in two ways: first, regularity condition was replaced by a weaker normality condition (see Example 1.1.2°), which was enough for using properties of the associated symmetric space (X, D). Second, property (MKA_3) used here is slightly weaker than property (MKA_1) used in [5] (see Example 2.4).

Using Lemma 2.3 we obtain the following

Corollary 3.3. *Let (X, d) be a normal abstract metric space satisfying condition (2.1). If a mapping $f : X \to X$ satisfies property (MKA_2), then f has a unique fixed point u, and for each $x \in X$, the sequence $\{f^n x\}$ of Picard iterations converges to u.*

An open question is whether the conclusion of Corollary 3.3 remains valid if (X, d) is a normal abstract metric space with the normal constant $k = 1$ and $f : X \to X$ satisfies property (MKA_2).

Remark 3.4. Taking $E = \mathbb{R}$ and $P = [0, +\infty)$ in Corollary 3.3 one obtains the classical Meir–Keeler result [1] in an easier way.

Finally, we note that the authors of [8] claim that their Meir–Keeler-type Theorem 3.2 is an extension of the mentioned Proposition 2.1 in [5]. However, the composition $\theta \circ d$ in their theorem is simply another cone metric d_1 and the space (X, d_1) and the mapping f satisfy all required conditions to apply Proposition 2.1 [5] and also our Theorem 3.1.

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